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Let  $(a_1, b_1)$  be any interval containing  $x_0$ , and let  $M_1, m_1$  be the bounds of the function on  $(a_1, b_1)$ ; then the function assumes all values between  $m_1$  and  $M_1$ . Take the case where  $m_1 < f(x_0) < M_1$ . Continue in this way where  $a_1, a_2, a_3, \dots$  approach  $x_0$ , and likewise  $b_1, b_2, b_3, \dots$ . Let  $M_i, m_i$  be the bounds of the function on  $(a_i, b_i)$ . Then both  $M_1, M_2, \dots$  and  $m_1, m_2, \dots$  approach  $f(x_0)$ . Suppose for instance that  $M_1, M_2, M_3, \dots$  do not approach  $f(x_0)$ , but have  $M > f(x_0)$  as greatest lower bound. Take  $k$ , where  $f(x_0) < k < M$ , then there is one value  $\bar{x}$  on  $(a_1, b_1)$  such that  $f(\bar{x}) = k$ . Let  $i$  be large enough that  $(a_i, b_i)$  does not contain  $\bar{x}$ ; then we must have  $M_i < k$ , or otherwise there would be a root of  $f(x) = k$  on  $(a_i, b_i)$ , which is impossible. But this contradicts the assumption that  $M$  is the greatest lower bound of  $M_1, M_2, \dots$ . Since  $M_1, M_2, \dots$  and  $m_1, m_2, \dots$  both tend to  $f(x_0)$  it follows that  $f(x)$  is continuous at  $x = x_0$ .

The cases other than that where  $m_1 < f(x_0) < M_1$  follow in a similar way.

**Corollary.** *The theorem is true if  $f(x)$  assumes any value at most  $n$  times,  $n$  a constant.*

## NOTE ON FUNCTIONS WHICH APPROACH A LIMIT AT EVERY POINT OF AN INTERVAL.

By E. W. CHITTENDEN, University of Illinois.

In the foregoing paper Captain Williams has discussed properties of functions which approach a limit at every point of an interval. It is the purpose of this note to present the following theorem:

**THEOREM.** *If a function  $f(x)$  has a limit  $f'(x)$  at every point  $x$  of an interval  $(a, b)$ , then for every positive number  $\sigma$ , however small, the number of points at which the measure of discontinuity (saltus) exceeds  $\sigma$  is finite, and the set of points at which  $f(x)$  differs from the continuous function  $f'(x)$  is at most enumerably infinite.*

At any point  $x$  of the interval  $(a, b)$  there is for any small positive number  $e$  an open interval (segment)  $S_{xe} = (x - h < x' < x + h)$  such that for any point  $x'$  in the segment, distinct from  $x$ ,

$$|f(x') - f'(x)| < \frac{e}{4},$$

and also, since  $f'(x)$  is continuous,

$$|f'(x') - f'(x)| < \frac{e}{4}.$$

Hence

$$|\delta(x')| = |f'(x') - f(x')| < \frac{e}{2}.$$

Therefore the oscillation of the function  $\delta(x)$  on the set obtained from  $S_x$  by omitting the point  $x$  is less than  $e$ .

Every point of the interval  $(a, b)$  is enclosed in a segment  $S_x$  except the points  $a$  and  $b$ , for which  $S_a = (a \leq x < a + h)$ ,  $S_b = (b - h < x \leq b)$ . (It is to be

understood that  $h$  depends on both  $x$  and  $e$ .) From the Heine-Borel theorem (see, for instance, Veblen and Lennes, *Infinitesimal Analysis*, p. 34) it follows that there exists for every  $e$  a finite set of points

$$a = x_1 < x_2 < x_3 \cdots, x_{m-1} < x_m = b$$

such that every point of the interval  $(a, b)$  belongs to some segment  $S_{x_i}$ , and that for every  $i$  ( $= 1, \cdots, m$ ) the oscillation of  $\delta(x)$  on  $S_{x_i}$  is, if we omit the point  $x_i$ , less than  $e$ . Hence if at any point  $x$  the measure of discontinuity of the function  $\delta(x)$ , and therefore of  $f(x)$ , exceeds  $e$ ,  $x$  is some one of the points  $x_i$ . Assign to  $e$  successively the values  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$  and denote the points corresponding to  $e = 1/n$  by  $x_{ni}$  ( $i = 1, 2, 3, \cdots, m_n$ ). The set of all points  $x_{ni}$  is enumerable and contains the set of all points at which the saltus of  $\delta(x)$  is positive, which must therefore be an enumerable set. The function  $\delta(x)$ , and consequently the function  $f(x)$ , is continuous for every point  $x$  not in the set of points  $x_{ni}$ , since such a point belongs (for every value of  $n$ ) to a segment  $S_{x_{ni}}$  on which the oscillation of  $\delta(x)$  is less than  $1/n$ .

Consider the classical example of a function continuous at the irrational points of an interval and discontinuous at every rational point. The function  $f(x) = 0$ , if  $x$  is an irrational point of the interval  $(0, 1)$ ,  $f(x) = 1/q$  if  $x = p/q$  ( $p$  and  $q$  relatively prime integers and  $p < q$ ). The function  $f'(x)$  exists and vanishes identically. Hence  $\delta(x) = -f(x)$ .  $f(x)$  is discontinuous on a dense enumerable set and possesses the maximum degree of discontinuity permissible under the theorem.

The argument of this note can be extended immediately by means of suitable changes of the terminology so as to apply to any abstract set admitting a definition of distance and the generalized Heine-Borel theorem.

## THE NINE-POINT CIRCLE OBTAINED BY METHODS OF PROJECTIVE GEOMETRY.

By H. N. WRIGHT, Whittier College.

Place a mirror of zero dimensions, but with a fixed direction  $d$  at a point  $A$ . Then any line  $a$  through  $A$  reflects into a line  $a'$  through the same point, and  $a'$

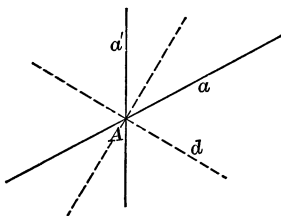


FIG. 1.

reflects back into  $a$ . Moreover by considering the angles of reflection it is clear that the pencil described by  $a$  is projective to the pencil described by  $a'$ . Thus